Then and now:
A world tour of nonlinear dynamics, stability, and chaos

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based on chapter of new book:

Rowena Ball[†] and Philip Holmes[‡], **Dynamical systems, stability, and chaos**. Chapter 1 in: J.P. Denier and J.S. Frederiksen (editors), *Frontiers in Turbulence and Coherent Structures.* World Scientific Lecture Notes in Complex Systems Vol. 6, 2007.

Download chapter from http://arxiv.org/abs/nlin/0702044v2

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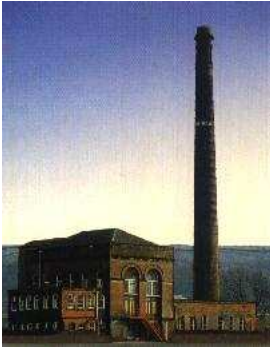
- Aspects of the mathematics of dynamical systems, stability, and chaos are reviewed within a historical framework that draws together the two major threads of its early development: **control theory** and **celestial mechanics**, and focusing on qualitative theory.
- Discussion of stability in two simple model problems.
- ~~Stability issues in a model for a magnetic fusion plasma.~~
- Recent extensions of stability theory to complex networks.

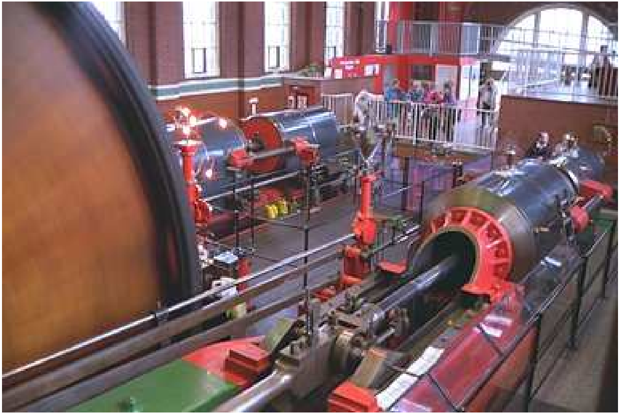
The tour begins

in a time and place where modern applied science and technology has its roots, in the industrial north of England. On the banks of a river near a village at the edge of the Lancashire Pennines we enter an imposing nineteenth century brick building. Here dwell two old ladies named Victoria and Alexandra.

They will never invite you in for tea though —

for the building is the Ellenroad Mill Engine House and the two Ladies are a giant, twin compound steam engine.





The speed of the engines is controlled by a centrifugal governor

This centrifugal flyball governor sits on a giant beam engine at Goulburn.

Patented by James Watt in 1789.

Most famous prototype example of a **self-regulating feedback mechanism**.

As engine speed increases the inertia of the flyballs swings the arms outwards, closing a valve which restricts the steam supply.

If the engine lags due to an additional load the flyballs are lowered and the valve opens, increasing the steam supply to compensate.

The disturbance itself actuates the restoring force.

Motion is fascinating to watch.

In certain operating regimes the motions of the governor may lose stability,

becoming oscillatory and spasmodic, amplifying the effect of the disturbance and thwarting control of the engine.

Nineteenth century engineers called this unstable behaviour *hunting* and devoted much effort to improving the design of centrifugal governors.

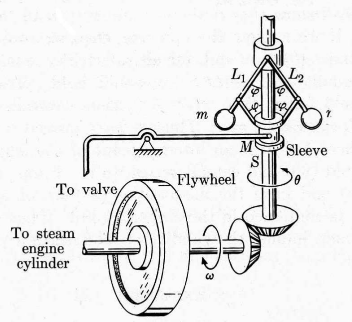
James Clarke Maxwell (1868)[1] was the first to formulate and analyse the stability of the equations of motion of the governor, followed (independently) by Vyshnegradskii (1876)[2].

[1] J. C. Maxwell. On Governors. Proc. Royal Soc. London, 16:270–283, 1868. Reprinted in: R. Bellman and R. Kalaba. Selected Papers on Mathematical Trends in Control Theory. Dover Publications New York, 1964.

[2] J. Vyshnegradskii. On the general theory of governors. (Sur la théorie génerale des régulateurs) . Comptes Rendus de l' Académie des Sciences de Paris, 83:318, 1876. (English translation in C.C. Bissell: Stodola, Hurwitz and the genesis of the stability criterion. Int. J. Control 50(6), 1989, 2313–2332).

Governor equations of motion: a case study

$$
\begin{aligned}
\frac{d\varphi}{dt} &= \psi \\
\frac{d\psi}{dt} &= n^2\omega^2 \sin\varphi\cos\varphi - g\sin\varphi - \frac{b}{m}\psi \\
\frac{d\omega}{dt} &= \frac{k}{J}\cos\varphi - \frac{F}{J},
\end{aligned}
$$



φ: angle between spindle S and flyball arms, $L = 1$, ω: rotational velocity of flywheel, transmission ratio $n = \theta/\omega$, θ: angular velocity of S, m: flyball mass, J: moment of inertia of the flywheel, F: net load on the engine, $k > 0$ is a constant, and b: frictional coefficient. For a given load F engine speed and fly-ball angle are required to remain constant, and the unique steady state is $\psi_0 = 0$, $\cos\varphi_0 = F/k$, $n^2\omega_0^2 = g/\cos\varphi_0$.

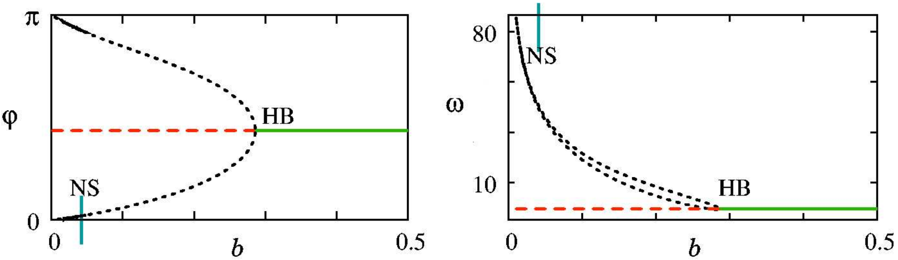
So far, so dull.

Dull, too, are the designers of engines, according to Maxwell:

“The actual motions corresponding to these impossible roots are not generally taken notice of by the inventors of such machines, who naturally confine their attention to the way in which it is designed to act; and this is generally expressed by the real root of the equation.”

The impossible roots Maxwell referred to are the complex roots of the characteristic equation obtained from the linearized equations of motion.

linear stability condition is $\frac{bJ}{m}\frac{\omega_0}{2F} > 1$.

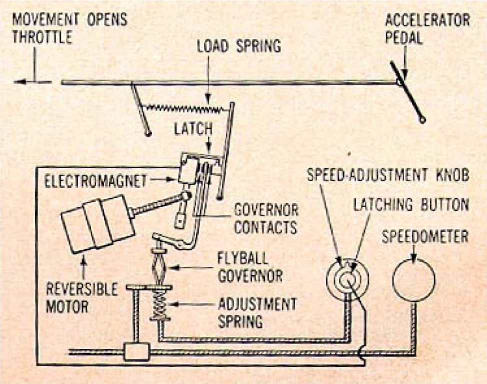


☛ Stable, steady state operation of the engine-governor system requires frictional dissipation b above the critical value at HB.
☛ Through NS the stable periodic solution undergoes a **Niemark-Sacker bifurcation:** a conjugate pair of Floquet multipliers leaves the unit circle and a two-dimensional asymptotically stable invariant torus bifurcates from the limit cycle. The behaviour of the system becomes essentially 3-dimensional.

The discovery of torus bifurcations first by Niemark (1959) in the USSR and independently by Sacker (1964) in the USA seems to be a classic case of unnecessarily duplicated development of mathematics during the cold war.

- The widespread adoption of the self-correcting centrifugal governor during the 18th and 19th centuries dramatically transformed the steam-driven textile mills, mining, and locomotion.
- Without it the incipient industrial revolution could not have progressed, because steam engines lacking self-control would have remained hopelessly inefficient, monstrous, contraptions, requiring more than the labour that they replaced to control them.
- Watt's iconic governor also embodies a radical change in the the paradigm of scientific culture.

the 1958 Chrysler Imperial brings control theory to the masses!



So control theory was a major strand in the development of modern nonlinear dynamics — but it was not the first. The centrifugal feedback governor also transformed the practice of astronomy, in that it enabled fine control of telescope drives and vastly improved quantitative observations.

We now focus on this earlier force in the development of dynamical systems and stability theory:

Celestial Mechanics

Next stop: The Kepler Museum in Regensburg, where Kepler is given cheek by his Muse.

An early 19th century relief which depicts Kepler unveiling the face of Urania, the Muse of astronomy, whereupon she coolly hands him a telescope and a scroll inscribed with his own laws, as if to say: "Hmm... not a bad job; now take these back and do some more work then tell me why your elliptical orbits are non-generic".

The one-dimensional Kepler ellipse, or the two-body problem of Newton, can be transformed into a harmonic oscillator with Hamiltonian

$$H(Q,P) = \frac{1}{8}P^2 - EQ^2.$$

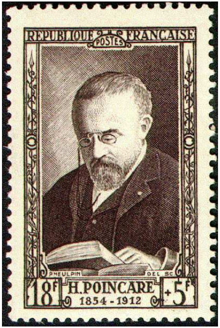
King Oscar's competition

- The problem of the stability of the solar system took a central role in the preoccupations of mathematicians, physicists, astronomers, and navigators post-Kepler.
- It was by no means clear, even to Newton, that Newton's law was sufficient to describe the motions of three or more celestial bodies under mutual gravitational attraction.
- Progress was made in the mid-1800s in improving series approximations, but the hydra of nonconvergence soon raised one after the other of its ugly (of course!) heads.
- By 1885, when it was chosen by Weierstrass as one of four problems in the mathematics competition sponsored by King Oscar II of Sweden, the n-body problem had achieved notoriety for its recalcitrance — but in doing so it had also driven many of the seminal advances in mathematics and produced many of the greatest mathematicians of the 19th century.

The first problem in King Oscar's competition

was to show that the solar system as modeled by Newton's equations is stable.

In his (corrected) entry Henri Poincaré (1890) invented integral invariants, characteristic exponents, and **Poincaré maps** (obviously), the **recurrence theorem**, proved the nonexistence of uniform first integrals of the three body problem, other than the known ones, discovered **asymptotic solutions** and **homoclinic points**, wrote the first ever description of chaotic motion — and founded and developed the entire subject of geometric and qualitative analysis.



He concluded by saying he regarded his work as only a preliminary survey from which he hoped future progress would result.

Poincaré's "preliminary survey" is still inspiring new mathematics and applications.

Poincaré's restricted three body problem:

Two massive bodies move in circular orbits on a plane with a third body of negligible mass moving under the resulting gravitational potential. A surrogate for this problem is a two degree of freedom Hamiltonian describing a simple pendulum coupled weakly to a linear oscillator:

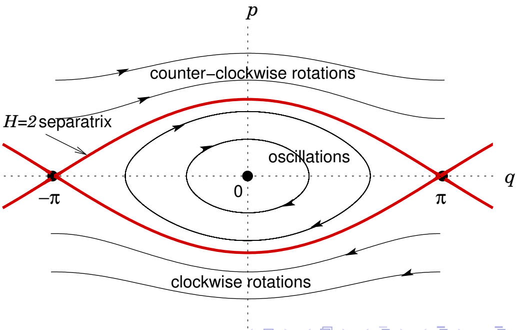
$$H(q_1, q_2, p_1, p_2) = -p_2 - p_1^2 + 2\mu \sin^2(q_1/2) + \mu\varepsilon \sin q_1 \cos q_2,$$

from which one obtains the reduced equations of motion

$$q_1' = -\partial P_h/\partial p_1 = 2p_1, \quad p_1' = \partial P_h/\partial q_1 = \mu \sin q_1 + \mu\varepsilon \cos q_1 \cos q_2,$$

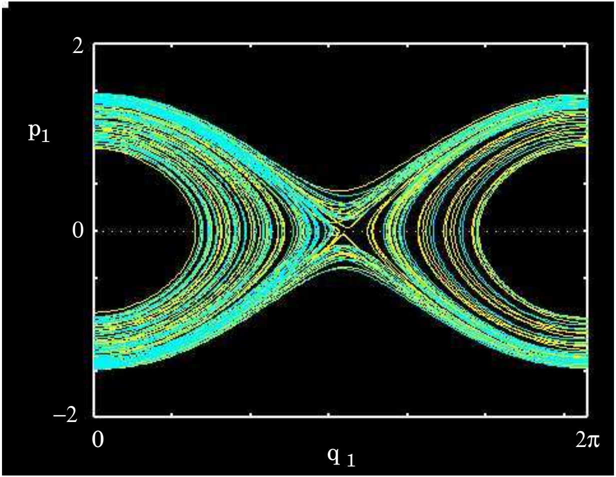
where $(\cdot)'$ denotes d/dq_2, so the angle variable q_2 plays the role of time.

For $\varepsilon = 0$ this is a simple pendulum:

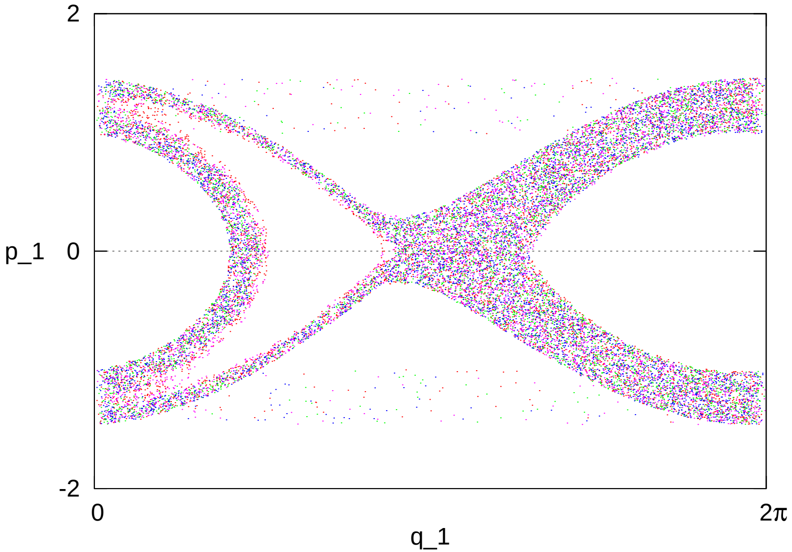


For $\varepsilon > 0$

...the separatrix splits so that orbits wander between librations and rotations, giving sensitive dependence and chaos.



Poincaré map of one orbit



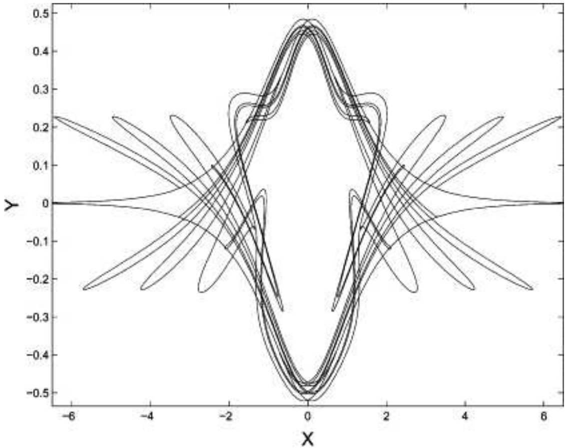
Poincaré decides that a thousand words are worth more than a picture

For $\varepsilon > 0$ the stable and unstable manifolds that form the separatrix level set typically break up, but some homoclinic points may persist.

"When we try to represent the figure formed by [the stable and unstable manifolds] and their infinitely many intersections, each corresponding to a doubly asymptotic solution, these intersections form a type of trellis, tissue or grid with infinitely fine mesh. Neither of the two curves must ever cross itself again, but it must bend back upon itself in a very complex manner in order to cut across all of the meshes in the grid an infinite number of times."

Thus did Poincaré (1899) describe homoclinic chaos.

Where angels fear to tread



Some transverse intersections of stable and unstable manifolds of the Poincaré map of a two-degree of freedom Hamiltonian corresponding to a driven harmonic oscillator.

R.H. Goodman, P.J. Holmes, and M.I. Weinstein. Interaction of sine-Gordon kinks with defects: phase space transport in a two-mode model. Physica D: Nonlinear Phenomena 161, 21–44, 2002.

$t = 0$

A cloud over Parliament House, Canberra,dissolves into homoclinic chaos!?

$t = 15$ *mins*

Calculations of homoclinic tangles in a numerical model of a tokamak magnetic field.
Mechanism for nondiffusive loss of particles.

R.K.W. Roeder, B.I. Rapoport, and T.E. Evans. Explicit calculations of homoclinic tangles in tokamaks. Physics of Plasmas 10, 3796–3799, 2003.

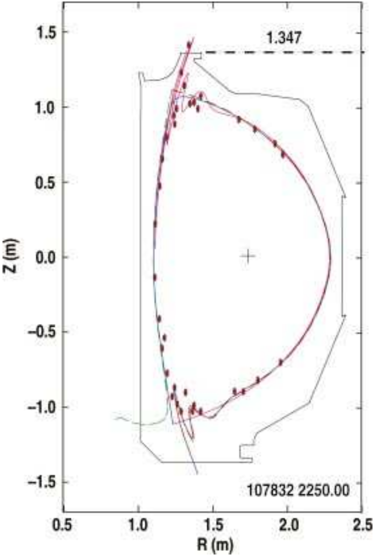


FIG. 3. (Color) Homoclinic tangle from the TRIP3D_MAP with a C-coil current of 10000 amp-turns. Large points plot a trajectory that escapes through the separatrix.

For the restricted three body problem itself, Poincaré showed that after truncating certain higher order terms in the expansion the Hamiltonian becomes completely integrable.

He also showed that the reduced system, and therefore its Poincaré map, possesses hyperbolic saddle points whose stable and unstable manifolds, being level sets of the second integral, coincide, as they do for the simple pendulum.

He then asked the key question in the qualitative approach to dynamical systems: **Should I expect this picture to persist if I restore the higher order terms**? In other words, is the reduced system **structurally stable**?

It is now known that integrable Hamiltonian systems of two or more degrees of freedom are not structurally stable.

- Control theory emphasizes Lyapunov stability, or local stability of solutions.
- ★ Celestial mechanics emphasizes structural stability, or the global behaviour of dynamical systems.

Bypassing several destinations on the world tour of nonlinear dynamics, including the Kurnell oil refinery, a beach in Rio, a street in Hanoi, and a rock art gallery in Carnarvon Gorge, we arrive at . . .

The 41st floor of an office tower in Hong Kong,

where the transport operations of Hong Kong, Kowloon, and the New Territories (or HKSAR) are controlled.

Stability of complex networks

The job of the HKSAR Department of Transport is formidable —

- The public transport network carries over **11 million** passenger trips **each day** and this number will increase.
- It consists of railways, franchised buses, public light buses, private buses, ferries, trams, and taxis. Each of these components is a complex sub-network in its own right.
- The area is geographically diverse, with islands, waterways, steep hills, and old built-up districts with limited road space.
- Environmental imperatives require the use of or conversion to low or zero emissions locomotive units.
- Efficient integration with transport in the densely populated economic-tiger zones of the Pearl River Delta is necessary.
- The network as a whole must be safe, affordable, reliable, and robust. It must minimize redundancy and duplication, yet be flexible enough to meet new and changing demands. This means it must be capable of response and adaptation on two time scales, daily and long-term (approximately yearly).

In dynamical systems language we ask: Is the HKSAR public transport network stable?

Intuitively (or through direct experience) we expect such a complex network to exhibit **sensitive dependence on initial conditions**: One blinking red LED on a signal-room console leads to a log-jam of peak hour trains.

Even with no perturbations on the network itself we know (with depressing certitude) that leaving for work five minutes later than usual is likely to result in arriving at work an hour late.

The effect of a perturbation on one element of a network seems to propagate as a growing instability through the network.

Naïvely, one expects that increasing the fraction of interacting elements or increasing the strength of interaction will enhance the stability of a complex network, but that is not necessarily so.

R. May (Nature, 238:413, 1972)

used random matrix theory to show that in a large, linear, randomly coupled network the system dimension and the coupling strength must together satisfy a simple inequality.

Work in progress: can this result be applied to manage a public transport network?

Consider $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ as the linearization of a (large) set of nonlinear first-order differential equations that describe rates of passenger turnover at each of n nodes in a public transport network.

The elements of $\mathbf{x}$ are the passenger numbers x_j at each node and the elements a_{jk} of the interaction matrix $\mathbf{A}$ describe the effect of node k on node j near equilibrium. Each a_{jk} is assigned from a statistical distribution of random numbers that has a mean of zero and a mean square value α, which expresses the average interaction strength.

The probability that any two nodes will interact is expressed by the connectance C, measured as the fraction of non-zero elements in $\mathbf{A}$.

For any given system of size n, average interaction strength α, and connectance C we ask:

What is the probability $P(n, \alpha, C)$ that any particular matrix drawn from the ensemble gives a stable system?

It can be shown that for large n the linearized system is almost certainly stable ($P(n, \alpha, C \to 1)$) if

$$\alpha < (nC)^{-1/2},$$

and almost certainly unstable ($P(n, \alpha, C \to 0)$) if

$$\alpha > (nC)^{-1/2}.$$

A transport network that is too richly connected (large C) or too strongly connected (large α) is likely to exhibit instability and that the effect is more dramatic the larger the number of nodes n. **This result is based firmly on stability theory as it was developed by Poincaré and Lyapunov.**

Predictions

Although dynamical systems, stability, and chaos theory were born and bred in celestial mechanics and control engineering, the concepts and methods now have much wider application to problems in new and developing fields of complex systems science. How will such problems be tackled?

1. Qualitative and asymptotic analysis. *(What is the complex systems analogue of the restricted three body problem that produced the most profound mathematical developments of the nineteenth century?)*
2. Interdisciplinary collaboration. *(Econophysics, biophysics, bioinformatics, climate change, social sciences. . .)*
3. Computation, *in silico* experiments, and visualization. *(The Geek approach, which involves terms such as GCC, MPI, Makefile, and people who know what http stands for.)*